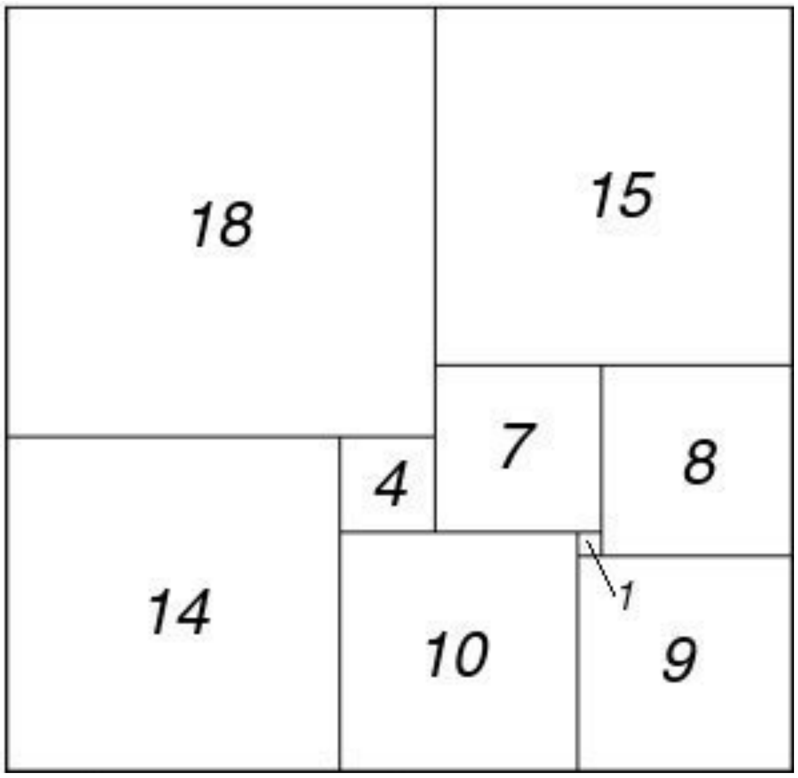
The Guest Column

Squaring the Square, by John E. Miller

Can you subdivide a square in to smaller squares? Obviously — a checkerboard divides a square into 64 squares. Likewise, you can divide an 8x8 unit square into one 4x4 square plus four 2x2 squares and thirty-two 1x1 squares — or any combination that covers the whole square. (1x4x4 + 4x2x2 + 32x1x1 = 64)

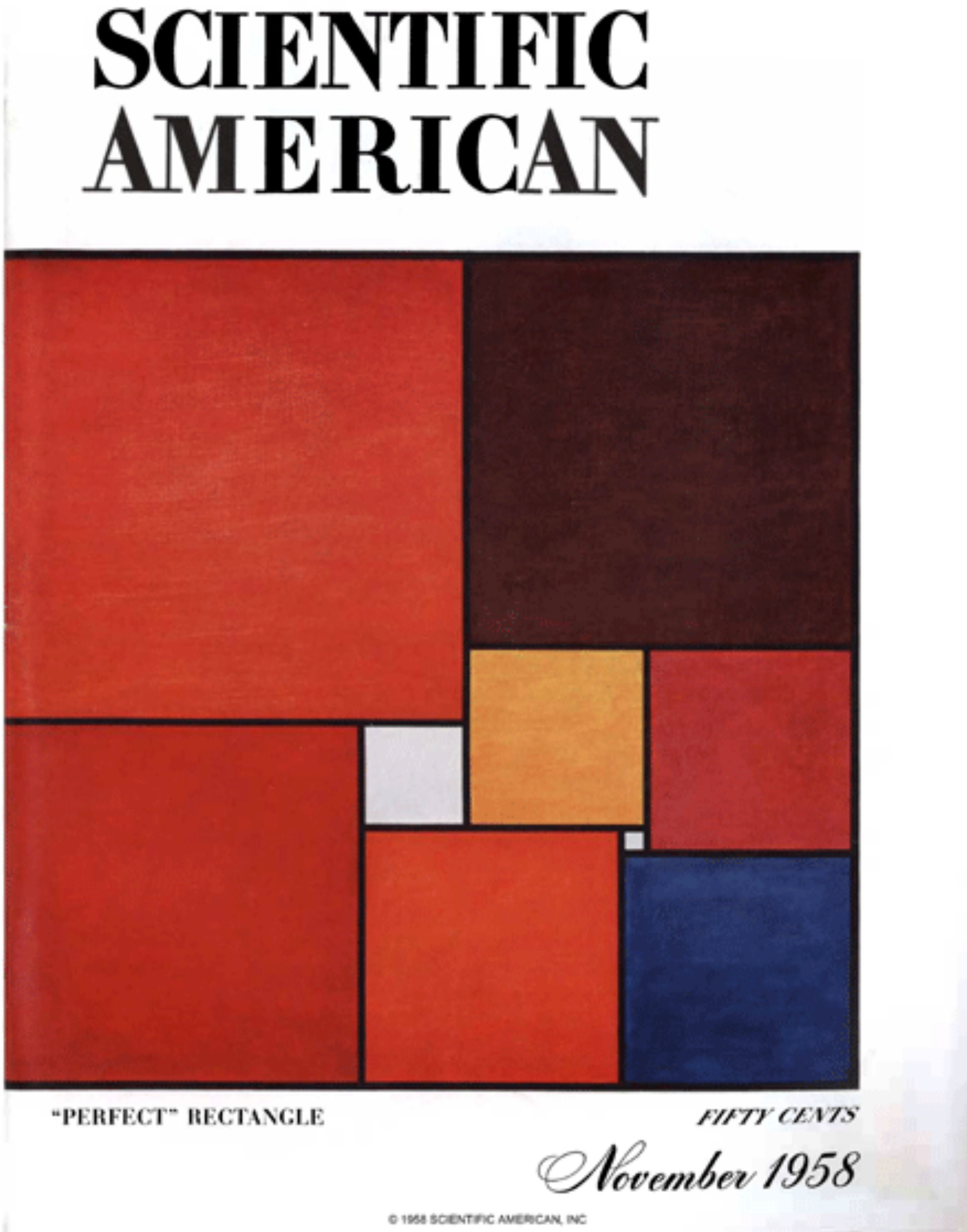
Now consider dividing a square into a number of squares, all with different (integer) side lengths! Prior to 1938, it was speculated that it likely wasn't even possible to dissect a square into unique squares. So at first, people were satisfied just to find that rectangles (such as this 32x33 nearly square rectangle) could be so divided.



This 32x33 rectangle is dissected into 9 squares.

Enter Martin

Martin's November 1958 column was titled: "How rectangles, including squares, can be divided into squares of unequal size". Martin introduced the problem in the first paragraph, but the rest of the column was literally quoted as presented by William Thomas Tutte. The cover of that issue of Scientific American drew attention to the column.



Scientific American cover "PERFECT" RECTANGLE

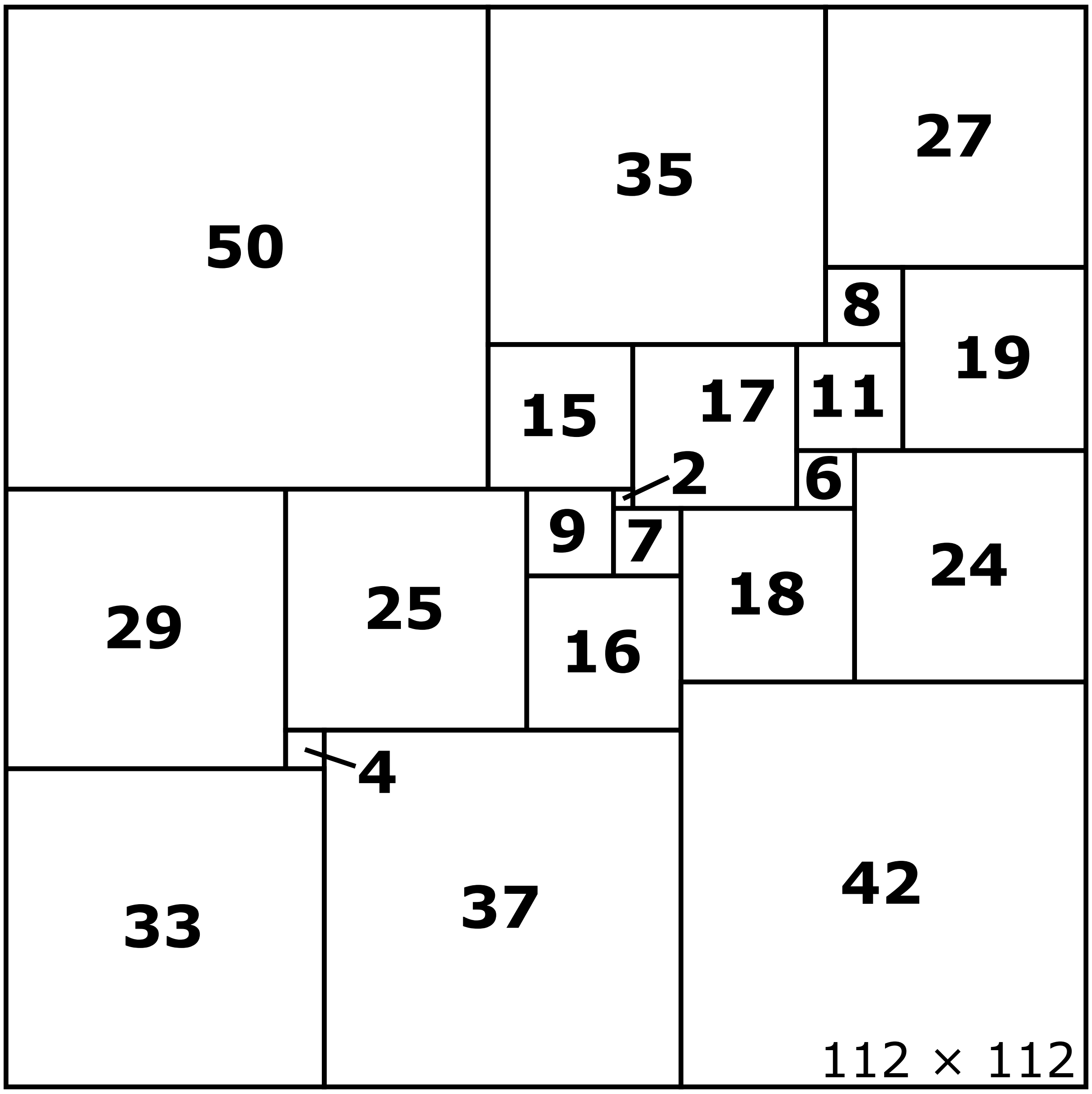
THE COVER is described on page 4: The painting on the cover, which resembles an abstraction by the artist Piet Mondriaan, is actually a representation of a "perfect" rectangle, that is, a rectangle made up of squares of unequal size (see "Mathematical Games", page 136).

The column explains how in 1939, electrical network theory was used to come up with solutions for filling rectangles and squares with squares. The first squared square to be found was a square filled with 69 smaller squares. The column ended with a challenge to find the smallest possible order perfect square.

In June 1978, some 40 years after the column appeared, Science and the Citizen in *Scientific American* reported that a 112x112 unit square had been dissected into just

21 different squares! It was previously known that there were no perfect squares of 20 or less, so when 21 was discovered by computer program, it was said to be a "Lowest Order Perfect Square". Not only that, it has also been proven to be unique — there are no other solutions for 21 squares. So this is a very unique mathematical object.

Martin references this 1978 report in the addendum in the chapter "Squaring the Square" of *The Second Book of Mathematical Puzzles and Diversions*. Here is the solution:



"Squaring the square" by N.Mori — . Licensed under Public domain via Wikimedia Commons.

The 'Important Members' of the Trinity Mathematical Society

Explanation from Wikipedia: It is first recorded as being studied by RL Brooks, CAB Smith, AH Stone and WT Tutte at Cambridge University. They transformed the square tiling into an equivalent electrical circuit — they called a "Smith diagram" — by considering the squares as resistors that connected to their neighbors at their top and bottom edges, and then applied Kirchhoff's circuit laws and circuit decomposition techniques to that circuit.

The reader is referred to Tutte's original paper (in References) for the interesting full story.

Squaring.Net, The ultimate website for Squares

Stuart E. Anderson maintains an excellent website about Squares, squaring.net and is an expert in optimizing the search algorithm. His paper *Compound Perfect Squared Squares of the Order Twenties* defines the terms used for squared square variants, gives an excellent history of the search and discovery of squared squares (1902-2013), gives computer models for solving CPSS problems, and catalogs known solutions using a special notation called Bouwkampcode, which encodes the arrangements of squares.

Squaring the Plane

In 1975, Solomon Golomb wondered whether the whole plane could be tiled by squares whose sizes are all natural numbers without repetitions. He called this the Heterogeneous Tiling Conjecture. This problem was later publicized by Martin Gardner in *Mathematical Games* and it appeared in several books, but it defied solution for over 30 years.

Recently, James Henle and Frederick Henle proved that this, in fact, can be done. (See reference at end.)

Puzzling the Square, by John Miller

When I was looking for ideas for interactive exhibits related to Martin for MoMath, I hit on the idea to make a puzzle based on the Simple Perfect Squared Square of Lowest Order — give people the 21 squares, and see if they can fit them into a large square frame. The frame would be just large enough to hold the assembled squares.

Before I made it, I worried that it would be too difficult, and a waste of time, etc. At first, I thought it might be too easy, because it seemed like once the six or so largest squares are in, the rest would literally fall into place (that is, they'd tell you where they go). Then I figured that it would probably be somewhere in between easy and hard, so I went ahead and made the squares in order to play with the "puzzle".

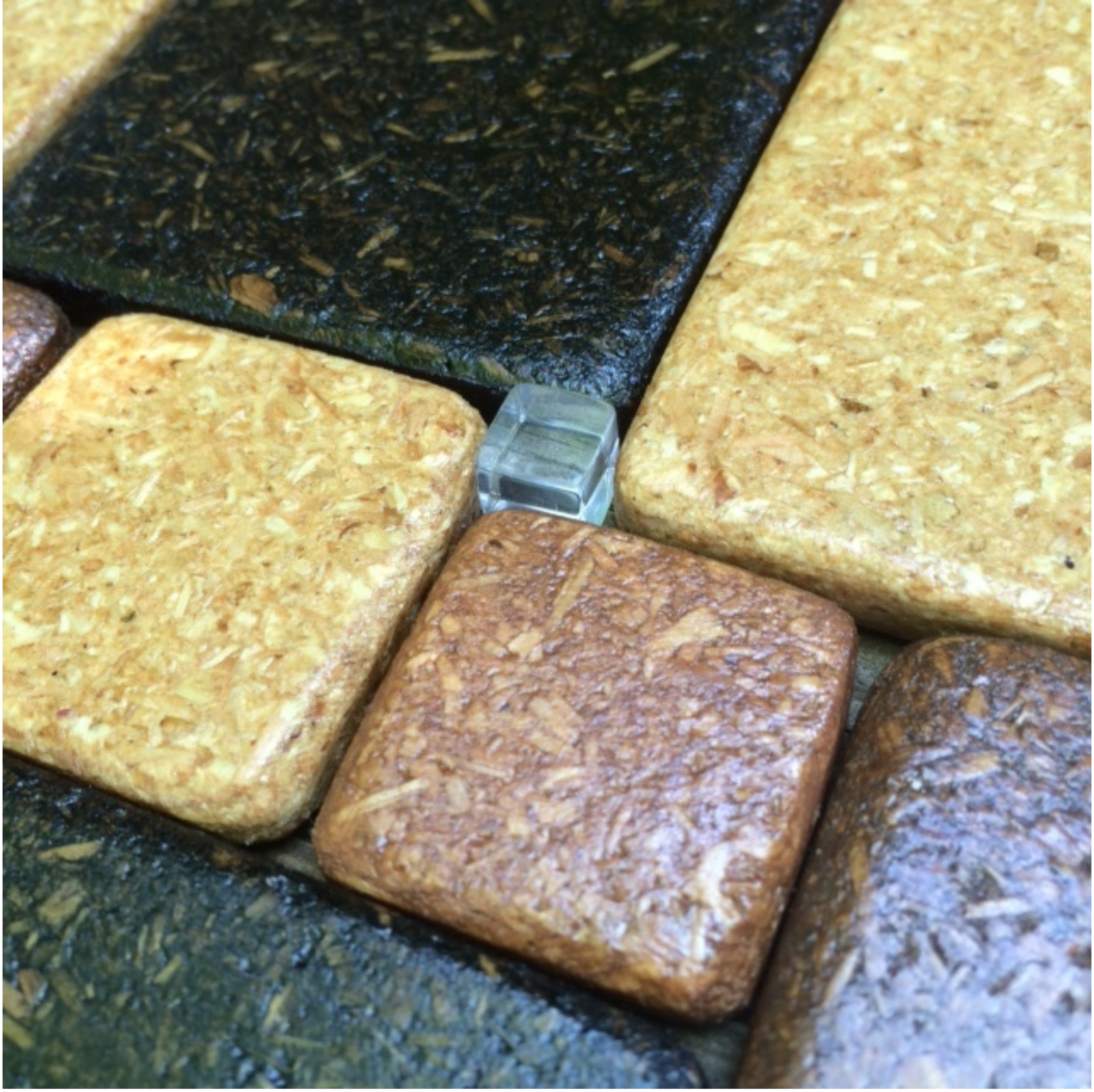
The Squared Square is 112 units. How large should a unit be in order to make a puzzle the overall size we might want? For the unit measure, I chose 3/16″ so that the whole puzzle would measure 21″x21″.

The 21 squares range from 50 units down to 2 units. Using 3/16″ units, the 2-unit square would measure 3/8″. The largest 50-unit square is 150/16″, or 9 3/8″. I simply used a ruler with numbered 3/16″ marks to measure off the squares.

I discovered a 4-coloring of the Squared Square on the internet (computed by Robert Harley) and used it to determine Stain the Squares. (Four Colors aren't strictly *needed* to solve the puzzle. A variation would be to paint the flip sides all black.)

The 21 square puzzle (working frame shown)

Since I chose to use 3/8″ particle board to make the squares, the 2-unit square was coincidently a cube!



The prize for solving the puzzle!

The puzzle was on display at the Martin Gardner Centennial booth at MathFest in Portland (August 6-9, 2014), where young and old came by to solve it. Since the 2-unit square was a small 3/8″ cube, I realized I could substitute a pre-made acrylic cube for the wood piece and give one away to each person who solved it. (We had a ready supply of plastic cubes under the table.)

Young MathFest Attendees Playing with Squares.

Using the Squared Square Pattern as Art

If I had to make another puzzle, I'd want to use a programmable laser cutting tool! --JM

Squared Square Patchwork Quilts, by Elwyn Berlekamp

The method for solving Squared Square problems provided a connection between passive network theory (which was previously well-known in EE circles and already considered a standard part of that "applied" curricula) and discrete combinatorial mathematics (which was then viewed as purely curiousity-driven or "recreational"). Claude Shannon and Peter Elias were both early advocates of building more connections between EE and discrete mathematics.

Claude Shannon's wife, Betty, weaved two quilt-like tapestries of squared squares. I'm not sure, but I think they were different from the current minimal one. She gave one of them to Peter Elias, a math PhD who was head of MIT's EE department in the early 1960s. It hung on the wall of his office, and provided a conversational opening (via Thevenin's and Norton's theorems) about the importance of moving more discrete math into the MIT's undergraduate EE curriculum.

MSRI borrowed that tapestry as part of their display during the inauguration of their new Simons' lecture hall in the late 00s. (Roger) Penrose gave the inaugural lecture, and somewhere I have a picture of me explaining that squared square to him. (We'd like to show that photo here! --ed)

Stained Glass

Anyone who works with stained glass needs no further instruction! Please send a photo of your project to the author. The best entry will be added here!


small project by author

Slate

You must allow for thickness of grout lines between squares — the geometry should run down the middle of all the grout lines. Send a photo or link please!

Showing your Squares?

We need a place for people to show off their projects. Suggestions please!

Contributors

John Miller, primary writer and puzzle maker. John's relation to Martin.

Elwyn Berlekamp, a co-counder of the G4G foundation. Elwyn's Home Page at Berkeley.

The Squares, and a Coloring by Robert Harley in France

The squares can be colored as follows so that no two adjacent squares have the same color.

Color A (39%) 6 squares {8, 9, 17, 24, 37, 50}
Color B (26%) 5 squares {2, 18, 25, 33, 35}

Color C (14%) 5 squares {6, 15, 16, 19, 29}
Color D (21%) 5 squares {4, 7, 11, 27, 42}

Terminology

A *squared rectangle* is a rectangle dissected into a finite number, two or more, of squares, called the *elements* of the dissection. If no two of these squares have the same size the squared rectangle is called *perfect*, otherwise it is *imperfect*. The *order* of a squared rectangle is the number of constituent squares. The case in which the squared rectangle is itself a square is called a *squared square*. The dissection is *simple* if it contains no smaller squared rectangle, otherwise it is *compound*.

A squared square which is both compound and perfect is called a compound perfect squared square (CPSS).

References

PAPER: *Simple Perfect Squared Square of Lowest Order*, A. J. W. DUIJVESTIJN, *Journal of Combinatorial Theory*, Series B 25, 240-243 (1978) [Where to download?]

WEB PAPER: Compound Perfect Squared Squares of the Order Twenties by Stuart E. Anderson is *the* definitive reference! This paper is also available via ARXIV.org. (Originally appeared in what journal?)

WEB SITE: squaring.net has all kinds of stuff, including a link to Tutte's original paper.

WEB PAGE: Catalog of Compound Perfect Squared Squared Squares (CPSSs); Orders 24 to 86

WEB PAGE: Brooks, Smith, Stone, Tutte (Part II) gives the technical *and* human story.

WIKIPEDIA PAGE: Squaring the Square. Also discusses Squaring the Plane and Cubing the Cube.

BOOK: Martin Gardner in the Twenty-first Century The chapter *Squaring the Plane* by Fred and James Henle discusses how they came to construct a method for "puffing up" an L-shaped region formed by two side-by-side and horizontally flush squares of different sizes into a perfect tiling of a larger rectangular region, then adjoining the square of the smallest size not yet used to get another, larger L-shaped region.

(Description from Wikipedia)

BOOK: *How Round is Your Circle? Where Engineering and Mathematics Meet*, John Bryant & Chris Sangwin, See 6.2 Duijvestijn's Dissection on page 114. Has B&W photo of the puzzle made of thin-veneered plywood, and a good discussion of the difficulty of fabricating exact puzzle pieces.
